J Eng Math (2008) 60:55–68
DOI 10.1007/s10665-007-9167-1

On the Navier–Stokes equations on manifolds with curvature

Marcelo H. Kobayashi

Received: 15 May 2005 / Accepted: 5 June 2007 / Published online: 27 July 2007

Abstract The main purpose of the present work is the development of a general formulation for the flow of a Newtonian fluid over manifolds with curvature. The novel formulation includes an explicit contribution of the Ricci curvature in the diffusion of momentum and is applicable to flows that conserve a general volume: Riemannian or otherwise. The general solution of the Stokes flow on a sphere and the associated Stokeslet are also computed.

Keywords Geophysical flows · Ricci curvature · Stokes flow on manifolds

1 Introduction

The present work concerns the extension of the Navier–Stokes (N–S) equation to include flows on a manifold with curvature. The N–S equation has known a long and illustrious history, starting with its derivation by Navier [1] and Poisson [2] from molecular considerations to Saint-Venant [3] and Stokes [4] who first provided the modern derivation of the (N–S) equations from the linear stress/rate-of-strain relation. In these momentous works the underlying physical space is tacitly assumed to be flat, which is the natural assumption for the study of the flow in our three-dimensional Euclidean space. More recently, in [5] (see also the references therein) an extension of the N–S equation was proposed for geophysical flows where the fluid is constrained to flow on a manifold. In that work, molecular diffusion is accounted for by the Bochner Laplacian operator applied to the velocity vector field.

The latter extension, while mathematically correct, is not the only rational way to formulate the N–S equation on a manifold. Indeed, Taylor [6] proposed a different extension of the N–S equation that starts with the strain-rate tensor and uses the non-commutativity of the second derivative in spaces with curvature to obtain a formulation of the N–S equation with the explicit inclusion of Ricci's curvature in the molecular diffusion. Taylor's equation agrees with the equation for surfaces previously proposed by Aris [7]. In our present derivation, we choose a different route: we extend the N–S equations from flat spaces to manifolds by first analyzing the motion of a Newtonian fluid on flow leaves, that is, on smooth surfaces of the Euclidean three-space, which are invariant by the fluid flow. These surfaces, which exist in any steady flow field, constitute physical examples of two-dimensional manifolds with a compatible fluid flowing in it. The cues from such a flow lead us to posit a novel formulation of the N–S equations,

M. H. Kobayashi (✉)
Department of Mechanical Engineering, University of Hawai'i at Manoa, 2540 Dole Street – Holmes Hall 302,
Honolulu, HI 96822, USA
e-mail: marcelok@hawaii.edu

where the effects of the manifold curvature are explicitly accounted for, as in Taylor's and Aris's work, but with a distinct diffusion term. Our extension of the N–S equation also allows for the preserved volume to be any general volume form, possibly different from the Riemannian volume, as is the case, for example, in axisymmetric flows.

Although not often used in fluid mechanics, Riemannian geometry is the germane theory for N–S equations on manifolds. Riemannian geometry is the backbone of the general theory of relativity [8], and it is also key in the study of classical mechanics [9–12] and control theory [13,14]. For fluid mechanics, the reader will find several interesting applications of Riemannian geometry in the comprehensive review [15]. Readers not familiar with Riemannian manifolds may consult [16,17].

The paper is organized as follows. In Sect. 2 we deduce the equation of motion of a viscous fluid in a flow leaf. Based on this equation, in Sect. 3 we propose a novel formulation of the N–S on manifolds and we solve the corresponding Stokes problem of the flow on a sphere. The solution to the Stokes equation would be useful for numerous applications including the determination of the forces acting on a particle that is moving on the surface of the Earth, or in the computation of the flow over a particle on a sphere under Marangoni convection. Some technical results are presented in the Appendix.

2 Navier–Stokes equations on flow leaves

In this section, we analyze the flow equations for the fluid on its flow leaves. Let M denote the manifold containing the fluid flow. We assume for the moment that $M = E^3$ is a three-dimensional Euclidean, affine manifold or $M = \mathbb{T}^3$ a three-dimensional flat torus. Suppose also the velocity field $u \in \mathcal{X}(M)$[1] is compatible with an integrable distribution $\mathcal{D} \subset TM$ of rank $= 2$, that is, $u(t, x) \in \mathcal{D}_x$, for all $t \in J$ and all $x \in M$, where J is the time interval where the solution is defined. We call the associated foliation $\Phi = (\mathcal{L}_i)_{i \in I}$ a *flow foliation* and the corresponding leaves $\mathcal{L} \in \Phi$ the *flow leaves*. In other words, the flow leaves are smooth surfaces in M to which the velocity field is tangent.

Remark 1 The use of the distribution $\mathcal{D}$ is not essential to our derivation. However, it simplifies some computations that require the extension of some vector fields, and on the other hand, it always exists (locally) in a steady flow. Moreover, the use of the distribution raises an intriguing question on the possibility of the solution to N–S equations being compatible with a non-integrable distribution.

Our first result is on the governing equation for the fluid on surfaces which are invariant by the fluid flow. Such surfaces can be obtained, for instance, by tracking a material line as it evolves in time in a steady flow or in flows with symmetries.

Proposition 1 *Assume the velocity field is compatible with the distribution $\mathcal{D}$. Then, any leaf $\mathcal{L} \in \Phi$ is invariant by the flow and for each leaf the equations governing the fluid motion on the leaf are*:

1. Continuity*:* $\mathsf{div}_{g_{\mathcal{L}}} u = \left(\mathrm{Tr} B_{\mathcal{D}^\perp}, u\right)_{g_{\mathcal{L}}}$,
2. Tangent N–S equations*:*
$$\frac{\partial u}{\partial t} + \nabla^\top_u u = -\frac{1}{\rho}\mathsf{grad}_{\mathcal{L}} p - \nu\left(\nabla^{\top *}\nabla^\top u + \mathsf{Ric}_{\mathcal{L}}(u) - \gamma^\sharp_\zeta\right) + P_{\mathcal{D}} b$$
3. Normal pressure gradient*:*
$$[B_{\mathcal{D}}(u,u)]^\flat + \mathsf{i}^*_{\mathcal{L}} * \left[\frac{\mathsf{d}p}{\rho} + \nu C^1_1(\nabla\zeta)\right] = 0,$$
where b is an external force field, ζ is the 2*-form vorticity field $\zeta = \mathsf{d}u^\flat$ and C^1_1 denotes the contraction of the first covariant index with the first contravariant* (*possibly raised*) *index.*

[1] We refer to the Appendix for the definitions and notation.

Proof Just use

$$\mathsf{d}\delta u^\flat(v) = \mathsf{d}_{\mathcal{L}}\delta_{\mathcal{L}}u^\flat(v) + \mathsf{d}_{\mathcal{L}}\left(\mathrm{Tr}B_{\mathcal{D}^\perp}, u\right)_{g_{\mathcal{L}}}(v) \tag{1}$$

for all $v \in T\mathcal{L}$ and $\delta_M \mathsf{d}u^\flat = \delta_M\zeta = -C_1^1(\nabla\zeta)$. □

Corollary 1 *Suppose* $\left(\mathrm{Tr}B_{\mathcal{D}^\perp}\right)^\flat$ *is exact and* $\zeta = \mathsf{d}u^\flat$ *is tangent to a leaf* $\mathcal{L} \in \Phi$. *Then, there exists a volume form* $\omega \in \Omega^2(\mathcal{L})$ *such that the equations of motion on the leaf* $\mathcal{L}$ *are*:

1. Continuity*:* $\mathsf{div}_\omega u = 0$,
2. Navier–Stokes equations*:*

$$\frac{\partial u}{\partial t} + \nabla_u^\top u = -\frac{1}{\rho}\mathsf{grad}_{\mathcal{L}}p - \nu\left(\nabla^{\top *}\nabla^\top u + \mathsf{Ric}_{\mathcal{L}}(u) - \left[\mathscr{L}_{\mathsf{grad}\log *\omega}u^\flat\right]^\sharp\right) + P_{\mathcal{D}}b.$$

3. Normal pressure gradient*:*

$$B_{\mathcal{D}}(u,u) + P_{\mathcal{D}^\perp}\frac{\mathsf{grad}p}{\rho} = 0.$$

The proof is given in the Appendix.

Flows on leaves that preserves a volume form other than the Riemannian volume are, for example, flows with a symmetry to rotations around an axis, or axi-symmetry. In this case, the flow leaves are semi-planes along the axis of symmetry and the two-dimensional flow on the plane leaves preserves the axi-symmetrical volume form $\omega = y\,\mathrm{d}x \wedge \mathrm{d}y$, where (x, y) are Cartesian coordinates in the plane, with x along the axis of symmetry, rather than the Riemannian volume $\omega_g = \mathrm{d}x \wedge \mathrm{d}y$.

The continuity and the N–S equations on the flow leaves have a physical meaning on their own. Indeed, these equations are written in coordinate-free form and depend only on the geometry and flow field on the manifold. They are similar to the corresponding equations in a flat space, but differ from them in fundamental ways: the effects of Ricci's curvature and the non-Riemannian volume change the viscous diffusion of momentum, and the divergence in the continuity equation should be taken, in general, with respect to a non-Riemannian volume.

3 Navier–Stokes on manifolds with curvature

The physical significance and completion of the continuity and N–S equations on flow leaves, as derived in Corollary 1, allow us to extend the continuity and N–S equations for an oriented Riemannian manifold in such a way that the former equations naturally agree with the governing equations on a flow leaf when the vorticity vector is orthogonal to the flow leaf.

The proposed general equations describing the motion of a Newtonian fluid with constant properties on a volume Riemannian manifold (M, g, ω) are then:

1. *Continuity equation*: $\mathsf{div}_\omega u = 0$,
2. *Navier–Stokes equation*:

$$\frac{\partial u}{\partial t} + \nabla_u u = -\frac{1}{\rho}\mathsf{grad}p - \nu\left(\nabla^*\nabla u + \mathsf{Ric}(u) - \left[\mathscr{L}_{\mathsf{grad}\log *\omega}u^\flat\right]^\sharp\right) + b.$$

3.1 Comparison with previous formulations

The formulations in [5] and [6] are only valid for the conservation of the Riemannian volume. Accordingly, the comparison below is provided in the context of a Newtonian fluid flow conserving the Riemannian volume. Clearly, the physical correctness of any of the previous formulations can only be determined by experimental evidence. However, it is worth noting the analytical differences incurred by the three formulations.

Let us start by writing, using the present notation, the viscous contribution (denoted $\mathcal{V}$) in the N–S equation for each formulation.

1. In Ref. [5] the viscous contribution is extended from the flat-space formulation by using the Bochner Laplacian:
   $\mathcal{V}(u) = -\nu\nabla^*\nabla u.$
   This extension corresponds to a mathematical extension of the operator used in the flat space to the general case of a manifold with curvature.
2. Taylor [6] in contrast starts with the constitutive law of the deviatoric stress tensor $\mathcal{S} = \nu(\nabla u + \nabla u^T)$. Taylor then takes the divergence of the deviatoric stress tensor and uses the non-commutativity of the second derivative in spaces with curvature and the incompressibility of the fluid to obtain:
   $\mathcal{V}(u) = \operatorname{div}\mathcal{S} = -\nu\left(\nabla^*\nabla u - \mathrm{Ric}(u)\right).$
3. In the present work we use the covariant formulation of the N–S equations and the extension to general manifolds is obtained from the analysis of the flow on the flow leaves. The results is:
   $\mathcal{V}(u) = -\nu\left(\nabla^*\nabla u + \mathrm{Ric}(u)\right).$

So the difference among the three formulations is the role the Ricci curvature plays in the diffusion of momentum. The first question regards the dissipative nature of the viscous term. In a Euclidean space, one important characteristic of the N–S equations is that the viscous term dissipates the total kinetic energy of the fluid. A simple calculation shows that all three formulations provide dissipative, even if at different rates, viscous terms (for the second formulation just note that $\mathcal{V}/\nu = -2\dot{\mathrm{D}}\mathrm{ef}^*\mathrm{Def}$, where $\mathrm{Def}(u) = 1/2(\nabla u + \nabla u^T)$ and recall that $\nabla^*\nabla$ is a symmetric, positive operator in the usual L^2 inner product of vector fields; as for the third formulation recall that $\Delta u = \nabla^*\nabla u + \mathrm{Ric}(u)$ and use the fact that Δ is a symmetric, positive operator with respect to the usual L^2 inner product of one forms).

So all three formulations above dissipate the total kinetic energy of the fluid. Locally, though, the behavior of the Ricci tensor can be very different in each formulation. Thus, the Ricci tensor has no explicit effect in the first formulation, regardless of the curvature of the manifold. As for the second and third formulation, the Ricci term provides a local force per unit of volume-measure that is aligned with the flow for manifolds with local negative curvature and a local stress opposing the flow in manifolds of local positive curvature.

To understand why this term produces such effects, let us consider the model of the fluid motion in a thin layer on a sphere "in current use in dynamical meteorology and oceanography" according to Batchelor in his celebrated book [18]. Indeed, in this book, Batchelor used four assumptions about the flow that leads to the neglect of the radial components of velocity and acceleration. With the assumption that the radial component of the velocity is negligible, we obtain the following equations for the viscous terms on a thin layer on a sphere, using the spherical coordinates (θ, ϕ) (see, for example, [18], *ibid.* p. 601):

$$\frac{\mathcal{V}_\theta^B}{\nu} = \frac{1}{R^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta u_\theta) + \frac{1}{R^2\sin^2\theta}\frac{\partial u_\theta}{\partial\phi^2} - \frac{u_\theta}{R^2\sin^2\theta} - \frac{2\cos\theta}{R^2\sin^2\theta}\frac{\partial u_\phi}{\partial\phi}, \tag{2}$$

$$\frac{\mathcal{V}_\phi^B}{\nu} = \frac{1}{R^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta u_\phi) + \frac{1}{R^2\sin^2\theta}\frac{\partial u_\phi}{\partial\phi^2} - \frac{u_\phi}{R^2\sin^2\theta} + \frac{2\cos\theta}{R^2\sin^2\theta}\frac{\partial u_\theta}{\partial\phi}, \tag{3}$$

where u_θ, u_ϕ are the physical components of the velocity vector field in the spherical coordinate system. Here $\mathcal{V}^B$ denotes the viscous term in the N–S equations on a thin spherical layer under the assumption that the radial component of the velocity is null.

In the present formulation, the viscous term is given by (compare equations (14)–(15)):

$$\frac{\mathcal{V}_\theta}{\nu} = \frac{1}{R^2}\frac{\partial}{\partial\theta}\left[\csc\theta\frac{\partial}{\partial\theta}\left(\sin\theta\, u_\theta\right)\right] + \frac{\csc^2\theta}{R^2}\frac{\partial^2 u_\theta}{\partial\phi^2} - \frac{2\cot\theta\ \csc\theta}{R^2}\frac{\partial u_\phi}{\partial\phi}, \tag{4}$$

$$\frac{\mathcal{V}_\phi}{\nu} = \frac{1}{R^2}\frac{\partial}{\partial\theta}\left[\csc\theta\frac{\partial}{\partial\theta}\left(\sin\theta\, u_\phi\right)\right] + \frac{\csc^2\theta}{R^2}\frac{\partial^2 u_\phi}{\partial\phi^2} + \frac{2\cot\theta\csc\theta}{R^2}\frac{\partial u_\theta}{\partial\phi}. \tag{5}$$

Manipulating the first term in Eq. 4 we obtain,

$$\frac{\partial}{\partial\theta}\left[\csc\theta\frac{\partial}{\partial\theta}\left(\sin\theta\, u_\theta\right)\right] = \frac{1}{R^2}\frac{\partial^2 u_\theta}{\partial\theta^2} + \frac{\cot\theta}{R^2}\frac{\partial u_\theta}{\partial\theta} - \frac{u_\theta}{R^2\sin^2\theta} = \frac{1}{R^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta u_\theta) - \frac{u_\theta}{R^2\sin^2\theta}.$$

An analogous result holds for Eq. 5. Thus we conclude that the proposed formulation agrees with the current models in meteorology and oceanography that assume negligible radial velocity. So, for example, using an eddy viscosity to simulate the turbulent flow in the thin layer and assuming null radial velocity would require the inclusion of Ricci's curvature as proposed here, besides other terms providing for the variations of turbulent viscosity in the original three-dimensional space. It is interesting to note that, under the previous conditions of null radial velocity, the other two extensions considered in the paper would not predict the same effect due to the curvature.

To understand why the curvature can create such local "push" or "pull" on the fluid particle, it is useful to recall its genesis. In any physical flow over a surface in the three-dimensional, Euclidean space, the stress acting over the fluid is proportional to the symmetric part of the velocity gradient. The viscous term in the N–S is then the divergence of this symmetric tensor. Even though the velocity vector is tangent to the surface, its gradient may have, in general, components in the normal direction. So, the divergence appearing in the equation of motion for a fluid flow tangent to the surface will also feel the variation of the component of the gradient of the velocity field in the normal direction to the manifold. In the sphere such a term appears as, for example, $-u_\theta/R^2\sin^2\theta$. In the general case, as shown in the paper, the physical quantity, independent of the coordinate system, associated with the divergence of this normal component of the velocity variation, is $-\mathsf{Ric}(u)$. Finally, to see why this term predicts a variation in the opposite sense of the Ricci curvature term we can use the Frenet apparatus which predicts that along a curve, say, the trajectory of the fluid particle, the variation of the normal vector parameterized by the arc length is proportional to a term involving the negative of the curvature of the curve times the tangent vector to the curve; in symbols:

$$N' = -\kappa T + \tau B,$$

where κ is the curvature of the curve, $B = T \times N$, and τ is the torsion vector. So, given the fact that the arc length is a reparametrization of the fluid-particle trajectory, and that the velocity vector is aligned with the tangent vector T, we deduce that the physical contribution to such a variation of the component normal to the surface where the flow takes place, must be against the curvature, and so, against the velocity vector for regions of local positive curvature, as in a sphere, and in the same sense of the velocity vector in local regions of negative curvature, such as in a hyperbolic surface.

3.2 Stokes equation on a manifold for Riemannian isochoric flows

As usual, the Stokes equation is obtained by letting the Reynolds number $\mathrm{Re} = U_0 L/\nu$ tend to zero, that is, $\mathrm{Re} \to 0$, while keeping the Strouhal number $\mathrm{Sl} = L\Omega/U_0$ finite. Here U_0, L, and Ω are characteristic velocity, length and frequency scales. The resulting equation is then

1. *Continuity equation*: $\mathsf{div}_g u = 0$,
2. *Stokes equation*:

$$-\,\mathsf{grad}\, p - \mu\left(\nabla^*\nabla u + \mathsf{Ric}(u)\right) = \rho b. \tag{6}$$

Let us consider first the homogeneous form of the Stokes equation, that is, when $b = 0$. From the covariant form of the homogeneous Stokes equation,

$$\mathsf{d}p + \mu\Delta u^\flat = 0 \tag{7}$$

it is easy to show that the pressure and the vorticity form are harmonic, that is,

$\Delta p = 0,$

and

$\Delta\zeta = 0.$

In fact, from Eq. 7 we conclude that pressure and vorticity are harmonic conjugates, in the sense that they satisfy a generalization of the Cauchy—Riemann equations:

$\mathsf{d}p = -\mu\delta\zeta,$

valid for any manifold—it is interesting to note that, for a two-dimensional (resp. three-dimensional) manifold, $M * \zeta$ is a function (resp. 1-form with three components), just as the imaginary part of a complex number (resp. quartenion) has one component (resp. three components).

This result provides an analogy between the velocity potential and the stream function in a potential flow, on the one hand, and the pressure and vorticity in a Stokes flow, on the other. For a potential flow, the fluid flow is a solution of the Euler equation which, in this case, represents the geodesic motion of the fluid in the group of volume preserving diffeomorphisms [19]. Accordingly, the velocity field plays a key role and the velocity potential and the stream function are both related to the velocity field. The Stokes flow, by contrast, represents an equilibrium equation for the stresses. So it is natural that the stresses play the main roles as harmonic conjugates: the pressure field representing the potential function of the isotropic component of the stress tensor, and the vorticity field representing the vector potential of the skew-symmetric part of the stress tensor.

Using Corollary 3 from the Appendix, the dynamics of the previous equations can be determined from:

$$\Delta^2\psi = 0. \tag{8}$$

To solve the non-homogeneous equation, we use a simple extension of Pozrikidis' [20] method to compute the Green function in flat spaces. The main difficulty in extending Pozridikis' method to arbitrary manifolds lies in the extension of the constant vector field multiplying the Dirac delta function. The natural choice would be to consider a parallel transport of the vector multiplying the Dirac delta function. However, the presence of the scalar curvature would complicate the computations. Instead, we propose a fully covariant version of Pozrikidis' method.

We seek a particular solution of the Stokes equation

$$\mathsf{d}p + \mu\Delta u^\flat = \rho b. \tag{9}$$

Let us start by solving the auxiliary equation

$$\Delta\Phi = \rho b. \tag{10}$$

Then, by applying δ to (9) and taking into account that δ and Δ commute, that is, $\delta\Delta = \Delta\delta$, we obtain

$\Delta p = \delta\rho b,$

so that we may choose

$p = \delta\Phi.$

Now, for $u^\flat$ we consider

$$u^\flat = \frac{1}{\mu}\delta\mathsf{d}H.$$

It follows immediately from the previous definition and $\delta^2 \equiv 0$ that $\mathsf{div}_g u = -\delta u^\flat = 0$, regardless of H. In other words, the continuity equation is satisfied for any choice of H.

Now to fix H we require $u^\flat$ to satisfy the Stokes equation (9) yielding

$\delta\mathsf{d}\Delta H = \Delta\Phi - \mathsf{d}\delta\Phi = \delta\mathsf{d}\Phi.$

And we may select H such that

$\Delta H = \Phi.$

3.3 Solution of the Stokes equation on a sphere

As an illustration of the computation of the Stokes flow on a manifold, we consider the flow on a sphere. Note that the only smooth flow on the whole sphere is the null flow. This assertion follows from two facts: (i) p being harmonic and so constant on the whole sphere, and (ii) $u^\flat$ being also harmonic (see Eq. 7, with p constant) and so null from the isomorphism $\mathcal{H}^1_g \cong \mathrm{H}^1(M)$ and the triviality of the group $\mathrm{H}^1(M)$ (the sphere is contractible). This fact contrasts with the infinite number of smooth solutions that are possible in the plane or the flat torus $\mathbb{T}^2$.

Thus, all non-null flows will have at least one singularity. To compute a general solution of the previous equation, we start by writing (8) in the spherical polar coordinate system (θ, ϕ):

$$x = R\sin\theta\cos\phi, \quad y = R\sin\theta\sin\phi, \quad z = R\cos\theta,$$

where (x, y, z) are Cartesian coordinates with origin in the center of the sphere and R is the radius of the sphere. Denote by Ψ the physical component of the stream function $\psi = \Psi R\sin^2\theta\, \mathsf{d}\theta \wedge \mathsf{d}\phi$. We have,

$$\left[\csc\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \csc^2\theta\frac{\partial^2}{\partial\phi^2}\right]^2 \Psi = 0.$$

The circularity of ϕ allows the expansion

$$\Psi(\theta, \phi) = \sum_{k\in\mathbb{Z}} \Theta_k(\theta)\mathrm{e}^{\mathrm{i}k\phi},$$

where Θ_k satisfies the following fourth-order, linear ODE:

$$\Theta_k^{(4)} + 2\cot\theta\,\Theta_k^{(3)} - [1 + (2k^2+1)\csc^2\theta]\Theta_k'' + (2k^2+1)\cot\theta\csc^2\theta\,\Theta_k' + k^2\left(k^2 - \cos 2\theta - 3\right)\Theta_k = 0.$$

The previous equation can be written in operator form as

$$L_k^2\Theta_k = 0, \tag{11}$$

where

$$L_k := \frac{\mathrm{d}^2}{\mathrm{d}\theta^2} + \cot\theta\frac{\mathrm{d}}{\mathrm{d}\theta} - k^2\csc^2\theta\,\mathsf{id},$$

with id the identity operator. To simplify the last operator, we rewrite it into a general coordinate $\chi = \chi(\theta)$:

$$L_k = \frac{\mathrm{d}^2}{\mathrm{d}\chi^2}\left(\frac{\partial\chi}{\partial\theta}\right)^2 + \left(\cot\theta(\chi)\frac{\mathrm{d}\chi}{\mathrm{d}\theta} + \frac{\mathrm{d}^2\chi}{\mathrm{d}\theta^2}\right)\frac{\mathrm{d}}{\mathrm{d}\chi} - k^2\csc^2\theta(\chi)\,\mathsf{id},$$

where χ is selected to simplify this equation: $\chi := \log[\cot(\theta/2)]$ and

$$L_k = \cosh^2\chi\left(\frac{\mathrm{d}^2}{\mathrm{d}\chi^2} - k^2\mathsf{id}\right).$$

Let us consider the case when $k \neq 0$. The two-dimensional kernel of L_k is then given by

$$\ker L_k = \left\{a\,\tan^k\left(\frac{\theta}{2}\right) + b\,\cot^k\left(\frac{\theta}{2}\right)\Big| a, b \in \mathbb{R}\right\}.$$

Any element of $\ker L_k$ is clearly a solution of (11); the remaining solutions can determined from

$$\cosh^2\chi\left(\frac{\mathrm{d}^2}{\mathrm{d}\chi^2} - k^2\mathsf{id}\right)\Theta_{k,\pm} = \mathrm{e}^{\pm k\chi}.$$

We can integrate the previous equation using the Laplace transform yielding,

$$\Theta_{k,+}(\chi) = \frac{\mathrm{e}^{k\chi}}{\cosh^2\chi} * \frac{\sinh k\chi}{k},$$

or

$$\Theta_{k,+}(\chi) = \frac{\cot^k\left(\frac{\theta}{2}\right)}{2k}\left[2kk!\cot^2\left(\frac{\theta}{2}\right){}_2\tilde{F}_1\left(1, k+1; k+2; -\cot^2\left(\frac{\theta}{2}\right)\right) - 1\right]$$
$$-\frac{\tan^k\left(\frac{\theta}{2}\right)}{2k}\left[k\left(F\left(\frac{k}{2}\right) - F\left(\frac{k+1}{2}\right)\right) + 1\right]$$

for $k > 0$, where ${}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) := \frac{{}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;z)}{\Gamma(b_1)\cdots\Gamma(b_q)}$ is the regularized hypergeometric function, and $F := \Gamma'/\Gamma$ is the digamma function, with Γ the gamma function. For $k < 0$ we can use the symmetry $\Theta_{k,+}(\chi) = \Theta_{-k,+}(-\chi)$. Finally, $\Theta_{k,-}(\chi)$ is computed from $\Theta_{k,-}(\chi) = \Theta_{-k,+}(\chi)$.

For $k = 0$ we have

$$\ker L_0 = \left\{a + b\log\left(\tan\left(\frac{\theta}{2}\right)\right)\middle| a, b \in \mathbb{R}\right\},$$

and the remaining solutions are:

$$\Theta_0 = \log\left(\cot\left(\frac{\theta}{2}\right)\right)\left[\log\left(\cot\left(\frac{\theta}{2}\right)\right) + \log(\sin\theta) - 4\log\left(\cos\left(\frac{\theta}{2}\right)\right)\right] - \mathrm{Li}_2\left(-\tan^2\left(\frac{\theta}{2}\right)\right),$$

where Li_2 is the dilogarithm function and

$$\Theta_0 = \log(\sin\theta).$$

To illustrate the Stokes flow over a sphere we plot in Figs. 1–3 the first 11 non-null basic flows associated with the first modes of the Fourier expansion, for $k = 0, 1, 2$. These figures show a gamut of diverse flow structures including flows with parallel streamlines (Fig. 1), twin vortices flows (Fig. 2, center right) and cellular flows (Fig. 2 center left, and right, and Fig. 3, center left and right). The flow in Fig. 1 left, with parallel velocity almost everywhere, would be particularly useful as the far-field flow for the computation of the flow over a sphere centered at $\tilde{\phi} = \tilde{\theta} = 0$. The plots depicting the flow for higher values of k—Fig. 3—show similar features to the previous figures but with finer structures of the flow along the longitude coordinate.

The method for the non-homogeneous Stokes equation is used to compute the Stokeslet solution for the flow on a sphere:

$$\mathsf{d}p + \mu\Delta u^\flat = \delta^{\mathrm{Dirac}}_{x_0,\alpha_0}, \tag{12}$$

where δ^{Dirac} is the Dirac delta 1-form defined as

$$\langle\delta^{\mathrm{Dirac}}_{x_0,\alpha_{x_0}}, \beta\rangle := \left(\alpha_{x0}, \beta_{x0}\right)_{\Omega^1(T_{x_0}M)},$$

for all 1-forms β.

To determine the Stokeslet, we first write the equation for the fundamental 1-form Φ

$$\Delta\Phi = \delta^{\mathrm{Dirac}}_{x_0,\alpha_0} \tag{13}$$

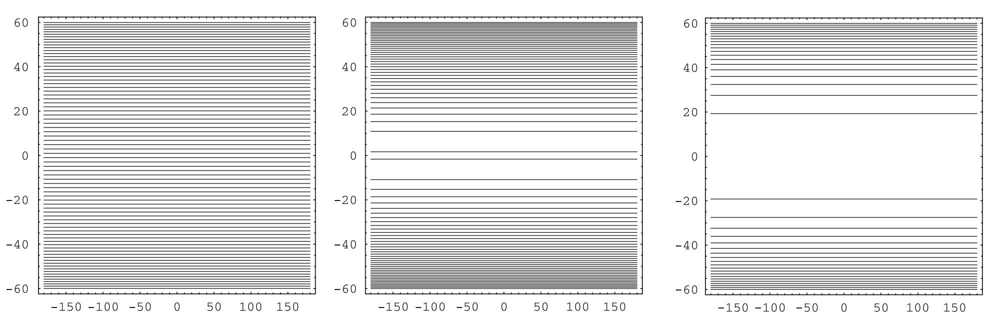

Fig. 1 Stokes solution for $k = 0$ in the $(\tilde{\phi}, \tilde{\theta})$ plane, with $\tilde{\phi} = \phi - \pi$ and $\tilde{\theta} = \theta - \pi/2$—units in degrees. Left: $\psi = \log\left(\tan\left(\frac{\theta}{2}\right)\right)$, center: $\psi = \log(\sin\theta)$ and right: $\psi = \log\left(\cot\left(\frac{\theta}{2}\right)\right)\left[\log\left(\cot\left(\frac{\theta}{2}\right)\right) + \log(\sin\theta) - 4\log\left(\cos\left(\frac{\theta}{2}\right)\right)\right] - \mathrm{Li}_2\left(-\tan^2\left(\frac{\theta}{2}\right)\right)$.

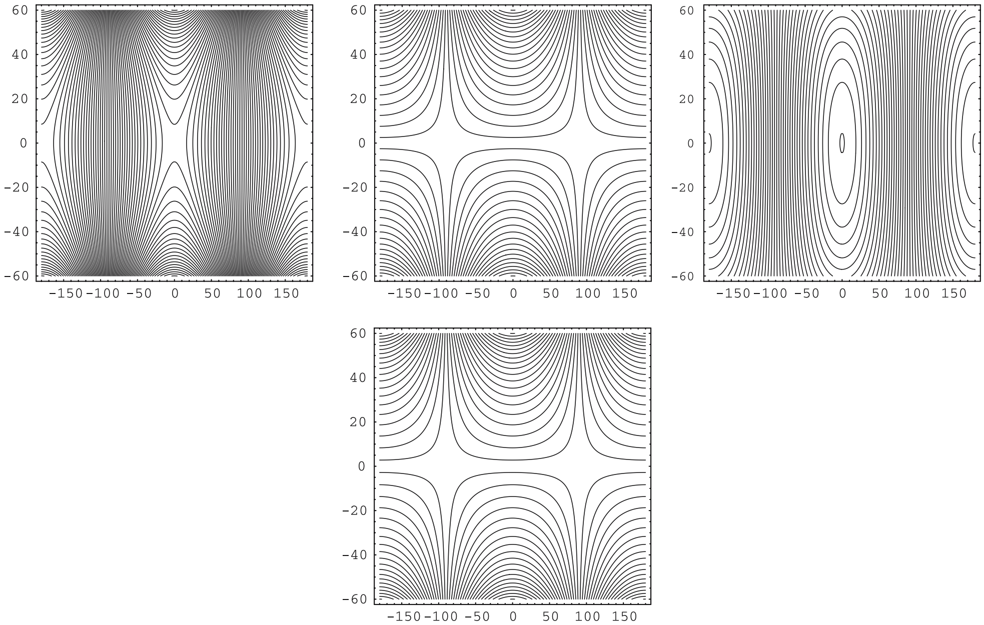

Fig. 2 Stokes solution for $k=1$ in the $(\tilde{\phi}, \tilde{\theta})$ plane—units in degrees. Left: $\psi = \csc\theta\cos\phi$, center-left: $\psi = \cot\theta\cos\phi$, center-right: $\psi = \csc\theta\left[(1+\cos\theta)\log\left(\cot\left(\frac{\theta}{2}\right)\right) - \log\left(\csc^2\left(\frac{\theta}{2}\right)\right)\right]\cos\phi$ and right: $\psi = \frac{\tan\left(\frac{\theta}{2}\right)}{2}\left[\left(\log\left(\sec^2\left(\frac{\theta}{2}\right)\right)+2\right)\cot^2\left(\frac{\theta}{2}\right) - \log\left(\csc^2\left(\frac{\theta}{2}\right)\right) - 2\right]\cos\phi$

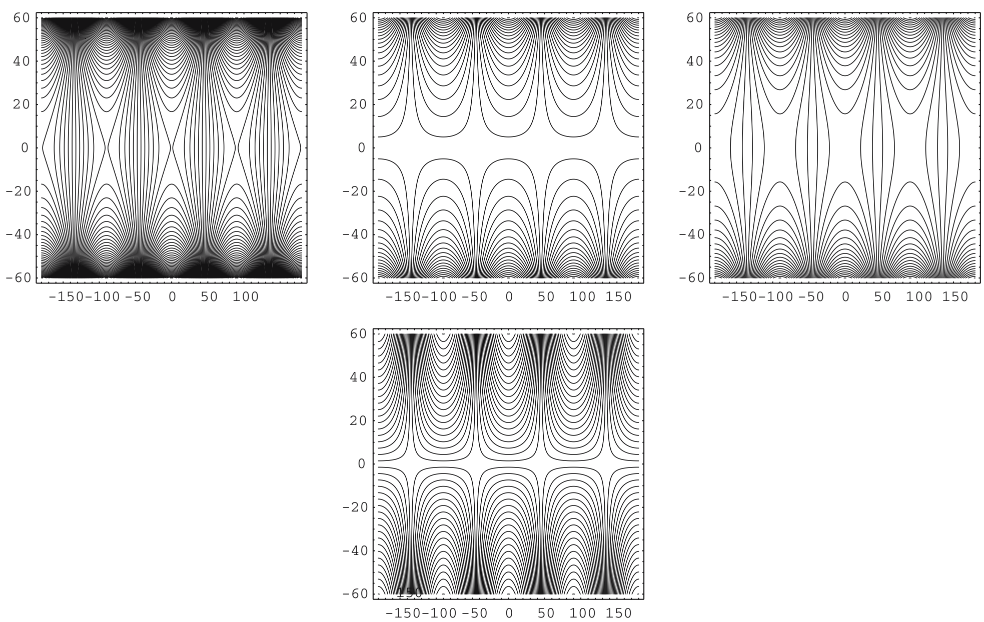

Fig. 3 Stokes solution for $k = 2$ in the $(\tilde{\phi}, \tilde{\theta})$ plane—units in degrees. Left: $\psi = \frac{1}{2}\left[\cot^2\left(\frac{\theta}{2}\right) + \tan^2\left(\frac{\theta}{2}\right)\right]\cos(2\phi)$, center-left: $\psi = 2\cot\theta\csc\theta\cos(2\phi)$, center-right: $\psi = \frac{1}{2}\left[\left(\log\left(\sec^2\left(\frac{\theta}{2}\right)\right)+1\right)\cot^2\left(\frac{\theta}{2}\right) + \left(\log\left(\csc^2\left(\frac{\theta}{2}\right)\right)+1\right)\tan^2\left(\frac{\theta}{2}\right) - 2\right]\cos\phi$ and right: $\psi = \frac{\tan^2\left(\frac{\theta}{2}\right)}{2}\left[\log\left(\csc^2\left(\frac{\theta}{2}\right)\right) - \cot^4\left(\frac{\theta}{2}\right)\log\left(\sec^2\left(\frac{\theta}{2}\right)\right)\right]\cos\phi$

in a neighborhood of x_0, using the polar coordinate system (θ, ϕ) above and such that $\alpha^\sharp_{x_0}$ is tangent to θ at $\theta = \pi/2$. We obtain:

$$\frac{\partial}{\partial\theta}\left[\csc\theta\frac{\partial}{\partial\theta}\left(\sin\theta\,\Phi_\theta\right)\right] + \csc^2\theta\frac{\partial^2\Phi_\theta}{\partial\phi^2} - 2\cot\theta\,\csc\theta\frac{\partial\Phi_\phi}{\partial\phi} = \frac{1}{\sin\theta}\delta^{\mathrm{Dirac}}_{\pi/2}\otimes\delta^{\mathrm{Dirac}}_{\phi_0}, \tag{14}$$

$$\frac{\partial}{\partial\theta}\left[\csc\theta\frac{\partial}{\partial\theta}\left(\sin\theta\,\Phi_\phi\right)\right] + \csc^2\theta\frac{\partial^2\Phi_\phi}{\partial\phi^2} + 2\cot\theta\csc\theta\frac{\partial\Phi_\theta}{\partial\phi} = 0, \tag{15}$$

where Φ_θ, Φ_ϕ are the physical components of Φ: $\Phi = \Phi_\theta R\mathrm{d}\theta + \Phi_\phi R\sin\theta\mathrm{d}\phi$. Without loss of generality we can take $\phi_0 = 0$. Owing to the circularity of ϕ we can expand the unknowns $\Phi_\theta = \sum_{n\in\mathbb{Z}}\hat{\Phi}_{\theta,n}\mathrm{e}^{\mathrm{i}n\phi}$ and $\Phi_\phi = \sum_{n\in\mathbb{Z}}\hat{\Phi}_{\phi,n}\mathrm{e}^{\mathrm{i}n\phi}$ and the Dirac distribution

$$\frac{1}{\sin\theta}\delta^{\mathrm{Dirac}}_{\pi/2}\otimes\delta^{\mathrm{Dirac}}_{0} = \sum_{l\in\mathbb{N}}\sum_{m=-l}^{l}(-1)^m Y_l^m(\theta,\phi)Y_l^{-m}(\pi/2,0),$$

where $Y_l^m(\theta,\phi) = \sqrt{\frac{(l-m)!}{(l+m)!}}\sqrt{\frac{2l+1}{4\pi}}P_l^m(\cos\theta)e^{\mathrm{i}m\phi}, l\in\mathbb{N}, m = -l,\ldots,l$ are the spherical harmonics and the polynomials

$$P_l^m(x) = \frac{(-1)^m}{2^l l!}(1-x^2)^{(m/2)}\frac{\mathrm{d}^{l+m}}{\mathrm{d}x^{l+m}}(x^2-1)^l,$$

are the associated Legendre polynomials.

After some manipulation we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left[\csc\theta\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\,\hat{\Phi}_n\right)\right] - n\csc\theta\,(n\csc\theta + 2\cot\theta)\,\hat{\Phi}_n = \frac{(2n+1)}{2^n n!4\pi}P_n^n(\cos\theta), \tag{16}$$

where $\hat{\Phi}_n := \hat{\Phi}_{\theta,n} + \mathrm{i}\hat{\Phi}_{\phi,n}$. We start with the solution of the homogeneous equation associated with (16):

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left[\csc\theta\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\,\bar{\Phi}_n\right)\right] - n\csc\theta\,(n\csc\theta + 2\cot\theta)\,\bar{\Phi}_n = 0. \tag{17}$$

Using the change of coordinate $\xi = \cos\theta$ we obtain

$$\bar{\Phi}_n''(1-\xi^2)^2 - 2\xi(1-\xi^2)\bar{\Phi}_n' - (1+n^2+2n\xi)\bar{\Phi}_n = 0, \tag{18}$$

where $' = \mathrm{d}/\mathrm{d}\xi$. Equation 18 has three regular singularities at -1, 1 and ∞ and so is of Hypergeometric type. The general solution of Eq. 17 for $n \le 0$ is then

$$\bar{\Phi}_n = \mathrm{e}^{n\tanh^{-1}\cos\theta}\csc\theta\left[a + b\frac{(\cos\theta+1)\,{}_2F_1\left(1-n,-n;2-n;\cos^2\left(\frac{\theta}{2}\right)\right)\sin^2\left(\frac{\theta}{2}\right)^{-n}\left(-\tan^2\left(\frac{\theta}{2}\right)\right)^n}{n-1}\right],$$

with $a, b \in \mathbb{R}$, and for $n > 0$ we can use the symmetry $\bar{\Phi}_n(\theta) = \bar{\Phi}_{-n}(\pi-\theta)$.

Now, using the variation of parameters it is easy to obtain a particular solution for the non-homogeneous equation (17) as:

$$\tilde{\Phi}_n = \frac{2n+1}{2^{2+n}\pi n!}\int_0^\theta \sin\sigma\tan^4\left(\frac{\sigma}{2}\right)\int_0^\sigma \log\left(\cot^4\left(\frac{\bar{\sigma}}{2}\right)\csc\bar{\sigma}\right)P_n^n(\cos\bar{\sigma})\sin\bar{\sigma}\exp(-2\tanh^{-1}\cos\bar{\sigma})\mathrm{d}\bar{\sigma}\mathrm{d}\sigma.$$

4 Conclusions

The main contributions of the paper are the derivation of an extension of the N–S equation that is based on the physics of fluids flowing over two-dimensional surfaces that are invariant by the flow in our Euclidean three-dimensional space. The derived equation satisfies the requirement of kinetic dissipation by the viscous term. The novel formulation also extends the N–S equations to flows that conserve general volume forms. The latter includes relevant flows such as flows with symmetry. The proposed equation is relevant for physical flows such as geophysical flows, Marangoni convection on surfaces, and two-dimensional turbulence in soap films (see e.g. [21]).

Also, the analytical solution of the Stokes problem has been explicitly derived. This solution would be interesting in the study of the flow over a particle moving on a sphere. That would provide new material for research in the three areas mentioned above.

Appendix A: Definitions and auxiliary results

Notation

M denotes a smooth, connected, paracompact manifold and $g\colon TM \to \mathbb{R}_+$ a smooth Riemannian metric on M, also represented as $g(w) = (w, w)_g$; TM (resp. T^*M) denotes the tangent (resp. cotangent) bundle of M and $\pi_{TM} : TM \to M, \pi_{T^*M}\colon T^*M \to M$ the associated projections. Smooth means C^∞ or real analytic (C^ω).∇ denotes the Levi-Civita covariant differentiation operator associated with the Riemannian manifold (M, g). $\mathcal{D}$ denotes a smooth distribution on M, that is, $\mathcal{D}$ is a vector sub-bundle of TM. With the distribution $\mathcal{D}$ and the metric g we have the smooth decomposition $TM = \mathcal{D} \oplus \mathcal{D}^\perp$ and the corresponding (smooth, VB-morphisms) projections $P_\mathcal{D}\colon TM \to \mathcal{D}$ and $P_{\mathcal{D}^\perp}\colon TM \to \mathcal{D}^\perp$.$\nabla^\top$ (resp. $\nabla^\perp$) will denote the projected covariant differentiation operator on $\mathcal{D}$ (resp. $\mathcal{D}^\perp$), that is, $\nabla^\top = P_\mathcal{D}\nabla$ (resp. $\nabla^\perp = P_{\mathcal{D}^\perp}\nabla$) and $B_\mathcal{D}\colon TM \oplus \mathcal{D} \to \mathcal{D}^\perp$ (resp. $B_{\mathcal{D}^\perp}\colon TM \oplus \mathcal{D}^\perp \to \mathcal{D}$) the associated total second fundamental form of $\mathcal{D}$ (resp. $\mathcal{D}^\perp$) [22], that is, given $u \in T_pM$, $v \in \mathcal{D}_p$ and $w \in \mathcal{D}_p^\perp$, then $(B_\mathcal{D}(u, v), w)_g = \left(\nabla_X Y, Z\right)_g$ where X, Y and Z are germs of smooth sections such that $X(p) = u$, $Y(p) = v$, $Z(p) = w$ and $X \in \mathcal{X}(M)$, $Y \in \Gamma(\mathcal{D})$, $Z \in \Gamma(\mathcal{D}^\perp)$. Let $N \subset M$ be a codimension 1 submanifold of M, then a k-form $\alpha \in \Omega^k(M)$ is parallel or tangent to N if the normal part, defined by $n\alpha = \mathsf{i}_N^*\,(*\alpha)$ is zero, where $\mathsf{i}_N\colon N \to M$ is the inclusion map. Analogously, α is perpendicular to N if its tangent part, defined by $t\alpha = \mathsf{i}^*\,(\alpha)$ is zero. Here $*\colon \Omega^k(M) \to \Omega^{n-k}$ denotes the Hodge star operator. Finally, the subscript g will denote variables associated with the metric. For example, div_g denotes the divergence operator corresponding to the Riemannian volume.

Definition 1 Let M be oriented and fix a volume form $\omega \in \Omega^n(M)$, where $n = \dim(M)$. We define the ω-codifferential operator $\delta_\omega : \Omega^{k+1}(M) \to \Omega^k(M)$ as

$$\delta_\omega\alpha = \frac{1}{*\omega}\delta[(*\omega)\alpha]$$

for all $\alpha \in \Omega^{k+1}(M)$, where $\delta := (-1)^{nk+1} * \mathsf{d}^*$, is the usual codifferential operator. Then, the ω- Laplace–deRhan operator $\Delta_\omega\colon \Omega^k(M) \to \Omega^k(M)$ is defined by $\Delta_\omega = \mathsf{d}\delta_\omega + \delta_\omega\mathsf{d}$. We call $\mathcal{H}_\omega^k(M) = \left\{\alpha \in \Omega^k(M) : \mathsf{d}\alpha = 0, \delta_\omega\alpha = 0\right\}$ the vector space of ω- harmonic fields.

Let M be compact and define the $\mathbb{L}_\omega^2$-inner product in $\Omega^k(M)$ to be

$$(\alpha, \beta)_{\mathbb{L}_\omega^2} = \int_M (\alpha \wedge *\beta)\,\omega,$$

for all $\alpha, \beta \in \Omega^k(M)$. Then, we can easily extend to ω-operators many results that are true for the case when ω is the Riemannian volume.

Proposition 2 *Let M be a compact boundaryless oriented Riemannian manifold, then the following holds:*

1. $\delta_\omega^2 = 0$,
2. $\delta_\omega = \delta - i_{\mathsf{grad}\log(*\omega)}$,
3. $\Delta_\omega = \Delta - \mathscr{L}_{\mathsf{grad}\log(*\omega)}$,
4. $\delta_\omega\left(X^\flat\right) = -\mathsf{div}_\omega X$, *for all* $X \in \mathcal{X}(M)$,
5. $\delta_\omega = \mathsf{d}^*$,

6. Δ_ω *is symmetric and elliptic,*
7. $\alpha \in \mathcal{H}^k_\omega(M)$ *if, and only if,* $\Delta_\omega \alpha = 0$,
8. ***Hodge Decomposition Theorem***:

$$\Omega^k(M) = \mathsf{d}\Omega^{k-1}(M) \oplus \delta_\omega \Omega^{k+1}(M) \oplus \mathcal{H}^k_\omega(M)$$

9. *The vector space* $\mathcal{H}^k_\omega(M)$ *is isomorphic to* $\mathrm{H}^k(M) = \ker \mathsf{d}^k / \mathrm{im}\, \mathsf{d}^{k-1}$.

Lemma 1 *Let* $X \in \Gamma(\mathcal{D})$ *then,* $\mathrm{Tr}\nabla(\mathsf{i}_{\mathcal{D}} \circ X)$= $\mathrm{Tr}\nabla^\top X$-$\left(\mathrm{Tr} B_{\mathcal{D}^\perp}, X\right)_g$, *where* $\mathsf{i}_{\mathcal{D}} : \mathcal{D} \to TM$ *is the inclusion map.*

Corollary 2 *If* $\mathcal{D}$ *is integrable with an associated foliation* $\Phi = (\mathcal{L}_i)_{i \in I}$ *and* $X \in \Gamma(\mathcal{D})$ *is a smooth section of* $\mathcal{D}$, *then over each* $\mathcal{L} \in \Phi$ *we have*

$$\mathsf{div}_g(\mathsf{i}_{\mathcal{D}} \circ X) = \mathsf{div}_{g_{\mathcal{L}}} X - \left(\mathrm{Tr} B_{\mathcal{D}^\perp}, X\right)_{g_{\mathcal{L}}}.$$

where $g_{\mathcal{L}}$ *is the induced Riemannian metric on the leaf* $\mathcal{L}$.

Note that from the fact that for any $\alpha \in \Omega(M)$, $\mathsf{d}\alpha = 2\,\mathrm{alt}(\nabla\alpha)$ it follows that the 1-form $(\mathrm{Tr} B_{\mathcal{D}^\perp})^\flat$ is exact, only if $\mathrm{alt}\left(\nabla(\mathrm{Tr} B_{\mathcal{D}^\perp})^\flat\right) = 0$. Also, if $\mathcal{D}$ is integrable and a leaf $\mathcal{L}$ is smoothly contractible, then $\mathrm{alt}\left(\nabla(\mathrm{Tr} B_{\mathcal{D}^\perp})^\flat\right) = 0$ is also sufficient for exactness. In this case, we can define a volume $\omega = \vartheta\mu \in \Omega^n(M)$, with $\vartheta = \exp(-f)$, where $d_{\mathcal{L}} f = \mathrm{Tr}\left(B_{\mathcal{D}^\perp}\right)^\flat$. And with this volume it is true that:

Proposition 3 *Let* $\mathcal{D}$ *be integrable and* $\mathcal{L} \in \Phi$ *be a smoothly contractible leaf such that* $\mathrm{alt}\left(\nabla(\mathrm{Tr} B_{\mathcal{D}^\perp})^\flat\right) = 0$, *then there exists a volume* $\omega \in \Omega^n(M)$ *such that,*

$$\mathsf{div}_g(\mathsf{i}_{\mathcal{D}} \circ X) = \mathsf{div}_\omega X$$

for all $X \in \Gamma(\mathcal{D})$.

In a flat space M, the equations describing the incompressible flow of a Newtonian fluid with constant properties are the well known continuity and N–S equations:

1. *Continuity*: $\mathsf{div}_g u = 0$, where $u \in \mathcal{X}(M)$ is the velocity vector field and
2. *Navier–Stokes equation*:

$$\frac{\partial u}{\partial t} + \nabla_u u = -\frac{1}{\rho}\mathsf{grad}\, p + \nu C^1_1(\nabla\nabla u) + b \tag{19}$$

where $\rho \in \mathbb{R}_+$ is the density, $\nu \in \mathbb{R}_+$ is the kinematic viscosity $p \in \mathcal{F}(M)$ is the pressure function, $b \in \mathcal{X}(M)$ is a given force field.

From the N–S equation (19), we derive the equation for the vorticity, which plays a relevant role in the equations for the flow leaves. We define the vorticity 2-form instead of the vorticity vector to simplify the computations in the sequel.

Proposition 4 *Let* $\zeta = \mathsf{d}u^\flat \in \Omega^2(M)$ *denote the vorticity form. Then,*

$$\frac{\partial \zeta}{\partial t} + \mathscr{L}_u \zeta + \nu\Delta\zeta = \mathsf{d}b^\flat$$

Given any orthonormal frame (ξ_1, ξ_2) of $\mathcal{D}_U$ for some coordinate neighborhood $U \subset M$, $N \in \Gamma(\mathcal{D}^\perp_U)$ & $\|N\| = 1$ and a 2-form $\eta \in \Omega^2(M)$ it is well defined the 1-form $v \in \mathcal{D}_U \mapsto \eta\left(\xi_\lambda, B\left(\xi_\lambda, v\right)\right) - \nabla\eta(N, N, v) \in \mathbb{R}$, $\lambda = 1, 2$. We define the section $\gamma_\eta \in \Gamma(\mathcal{D}^*)$ such that (locally) we have,

$$\gamma_{\eta_U}(v) = \eta\left(\xi_v, B\left(\xi_v, v\right)\right) - \nabla\eta(N, N, v) + \mathsf{d}_{\mathcal{L}}\left(\mathrm{Tr} B_{\mathcal{D}^\perp}, v\right)_{g_{\mathcal{L}}} + i_{\mathrm{Tr} B}\eta$$

for all $v \in \mathcal{D}_U$.

Corollary 3 *If in addition to the hypothesis of corollary 1 the leaf $\mathcal{L}$ is compact, boundaryless and smoothly contractible, then there exists a $\psi \in \Omega^2(\mathcal{L})$ such that*

$$u^\flat = \delta_\omega \psi,$$

and

$$\frac{\partial \Delta_\omega \psi}{\partial t} + \mathscr{L}_{\delta_\omega \psi^\sharp} \Delta_\omega \psi + \nu \Delta_\omega^2 \psi = \mathsf{d}_{\mathcal{L}} b^\flat.$$

Moreover, $(\mathcal{L}, \omega)$ is a simplectic manifold and $u \in \mathcal{X}(\mathcal{L})$ is a Hamiltonian vector field with Hamiltonian function $H = (\omega, \psi)_{g\mathcal{L}}$, that is,

$$i_u \omega = \mathsf{d}_{\mathcal{L}} H$$

Proof The first assertion is an immediate consequence of the Hodge decomposition theorem and (9) of Proposition 2. To prove the second it is enough to write the Navier–Stokes in its “covariant form”, that is,

$$\frac{\partial u^\flat}{\partial t} + \mathscr{L}_u u^\flat - \frac{1}{2} \mathsf{d}_{\mathcal{L}} \|u\|^2 + \nu \Delta_\omega u^\flat + \frac{1}{\rho} \mathsf{d} p = b^\flat;$$

then apply the exterior derivative to it, bearing in mind that the exterior derivative commutes with the Lie derivative and the ω-Laplace–deRhan operator,

$$\frac{\partial \zeta^\flat}{\partial t} + \mathscr{L}_u \zeta^\flat + \nu \Delta_\omega \zeta = \mathsf{d} b^\flat,$$

and observe that $\zeta = \mathsf{d} u^\flat = \mathsf{d} \delta_\omega \psi + \delta_\omega \mathsf{d} \psi = \Delta_\omega \psi$.

As for the remaining assertion,

$$\begin{aligned} *\,(i_u \omega) &= -(*\omega) u^\flat \\ &= -(*\omega) \delta_\omega \psi \\ &= -\delta(*\omega \psi) \\ &= *\mathsf{d}_{\mathcal{L}} *(*\omega \psi) \\ &= *\mathsf{d}_{\mathcal{L}} (\omega, \psi)_{g_{\mathcal{L}}}, \end{aligned}$$

and the result follows from the fact that the Hodge star operator is an isomorphism. □

Definition 2 The form $\psi \in \Omega^2(\mathcal{L})$ in Corollary 3 is called the stream form and $*\psi \in \mathcal{F}(\mathcal{L})$ is the stream function.

A.1 Proof of Corollary 1

The proof of the continuity equation is the same as that for Proposition 3, where a sufficient condition for exactness is also provided. To prove the remaining assertions we first note that the vorticity form being tangent to $\mathcal{L}$ (locally, for some coordinate neighborhood U) means

$$\zeta_U = \zeta_{12} \omega^1 \wedge \omega^2.$$

So, from a simple computation

$$\delta_U \zeta_U = \left(\frac{\partial \zeta_{12}}{\partial \hat{\omega}^2} - \zeta_{12} \Gamma_{33}^2 \right) \omega^1 - \left(\frac{\partial \zeta_{12}}{\partial \hat{\omega}^1} - \zeta_{12} \Gamma_{33}^1 \right) \omega^1 - \zeta_{12} \left(\Gamma_{12}^3 - \Gamma_{21}^3 \right) \omega^3.$$

Integrability of $\mathcal{D}$ implies $\mathrm{alt}(B_{\mathcal{D}}) = 0$, so the last term on the r.h.s. of the previous equation vanishes. Also, defining the function $\vartheta \in \mathcal{F}(U)$ by $\omega_U = \vartheta \omega^1 \wedge \omega^2$ exactness of $\mathrm{Tr} B_{\mathcal{D}^\perp}$ yields

$$\Gamma_{33}^\lambda \omega^\lambda = -\mathsf{d}_U \log \vartheta$$

whence

$$\delta_U \zeta_U = \delta_{\omega_U} \zeta_U$$

and from the continuity equation and Proposition 2, item (4) we obtain

$$\Delta_U u_U^\flat = \Delta_{\omega_U} u_U^\flat .$$

Then, the result follows at once from item (3) of Proposition 2 and the Weitzenbock formula [6]:

$$\Delta u = \nabla^* \nabla u + \mathsf{Ric}(u).$$

References

1. Navier LMH (1822) Memoire sur les Lois du Mouvement des Fluides. Memoires de l'Academie Royale des Sciences 6:389
2. Poisson SD (1831) Memoire sur les Equations Generales d'Equilibre et du Mouvement des Corps Solides Elastiques et des Fluides. Journal de l'Ecole Polytechnique 13:1
3. de Saint-Venant J-CB (1843) Note a Joindre au Memoire sur la Dynamique des Fluides. Comptes-rendus hebdomadaires des Seances de l'Acadamie des Sciences 17:1240
4. Stokes GG (1845) On the theories of the internal friction of fluids in motion, and of the equilibrium and motion of elastic solids. Trans Cambridge Philos Soc 8:287–305
5. Temam R (1988) Infitite dimensional dynamical systems in mechanics and physics. Springer-Verlag, New York
6. Taylor ME (1996) Partial differential equations III—nonlinear equations, vol. 117 of Applied Mathematical Sciences. Springer Verlag.
7. Aris R (1989) Vectors, tensors and the basic equations of fluid mechanics. Dover
8. Einstein A (1916) Die Grundlage der allgemeinen Relativitätstheorie. Annalen der Physik 49(7):769–822
9. Godbillon C (1969) Géométrie Différentielle et Mécanique Analytique, Collection Méthodes. Hermann, Paris
10. Arnold VI (1989) Mathematical methods of classical mechanics, 2nd edn, Graduate Texts in Mathematics. Springer Verlag, New York
11. Abraham R, Marsden JE (1978) Foundations of mechanics, 2nd edn. Addison-Wesley Publishing Company, Reading, Massachusetts
12. Oliva WM (2002) Geometric Mechanics, vol 1798 of Lecture Notes in Mathematics. Springer-Verlag, Berlin
13. Sastry S (1999) Nonlinear systems: analysis, stability and control, vol 10 of Interdisciplinary Applied Mathematics. Springer Verlag
14. Bonnard B, Chyba M (2003) Singular trajectories and their role in control theory, vol 40 of Mathematiques and Applications. Springer Verlag
15. Kambe T (2002) Geometrical theory of fluid flows and dynamical systems. Fluid Dyn Res 30:331–378
16. Abraham R, Marsden JE, Ratiu T (1988) Manifolds, tensor analysis, and applications, 2nd edn, vol 75 in Applied Mathematical Sciences. Springer-Verlag, New York
17. Lang S (1995) Differential and Riemannian Manifolds, 3rd edn, vol 160 of Graduate Texts in Mathematics. Springer Verlag, New York
18. Batchelor GK (1967) An Introduction to fluid dynamics. Cambridge University Press, Cambridge
19. Arnold VI, Khesin BA (1998) Topological methods in hydrodynamics, vol 125 of Applied Mathematical Sciences. Springer Verlag
20. Pozrikidis C (1992) Boundary integral and singularity methods for linearized viscous flow. Cambridge Texts in Applied Mathematics. Cambridge University Press
21. Rivera M, Vorobieff P, Ecke RE (1998) Turbulence in flowing soap films: velocity, vorticity, and thickness fields. Phys Rev Lett 81:1417–1420
22. Kupka I, Oliva WM (2001) The non-holonomic mechanics. J Differ Eq 169:169–189